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THE ESSENTIAL NORM OF AN OPERATOR IS NOT SELF-DUAL

BY

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ABSTRACT

It is established by an example that the distance of a bounded linear operator S from the class of compact operators on a Banach space is not always uniformly comparable with that of its adjoint S'. This provides a negative solution to an old problem. It is also shown that the seminorms due to Schechter and Weis, that measure the deviation from strict singularity or strict cosingularity of an operator, are not uniformly comparable with the corresponding distance functions. Both results rely on a general construction related to certain approximation properties that are associated with closed ideals of operators.

Introduction

Let E and F be Banach spaces. The bounded linear operator $R \in L(E,F)$ from E to F is compact, if the image RB_E of the closed unit ball B_E of E is relatively compact in F. The collection of compact operators from E to F is denoted by $K(E,F)$. A well-known theorem of Schauder states that $R \in K(E,F)$ if and only if its adjoint $R' \in K(F',E')$. The essential norm of $S \in L(E,F)$ is the quotient norm modulo the compact operators,

$$\|S\|_K = \inf\{\|S-R\|: R \in K(E,F)\}.$$

The inequality $\|S'\|_K \leq \|S\|_K$ follows from Schauder's result. It is an old problem to determine if the essential norm is self-dual in the sense that there is a constant $c > 0$ (possibly depending on E and F) such that

$$c\|S\|_K \leq \|S'\|_K \tag{0.1}$$

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for all $S \in L(E,F)$. This fundamental question is so natural, that it is difficult to pinpoint its occurrences in the literature. It was at least considered by Goldenstein and Markus [GM, pp. 51–52], Axler, Jewell and Shields [AJS] and Dilworth [D]. The principle of local reflexivity implies the related estimates

$$\frac{1}{5}\operatorname{dist}(S, A(E,F)) \le \operatorname{dist}(S', A(F',E')) \le \operatorname{dist}(S, A(E,F)) \tag{0.2}$$

for $S \in L(E,F)$, cf. [ET, Prop. 2], where $A(E,F)$ denotes the uniform closure of the class of finite rank operators in $L(E,F)$.

The main result of this paper (Example 2.5) exhibits Banach spaces E and F, where (0.1) fails to hold in $L(E,F)$ for any constant $c > 0$. The behaviour of the essential norm thus differs from that of (0.2). The example exploits the differences between certain approximation properties. The starting point is a quantitative characterization of a special compact approximation condition for Banach spaces. Results by Samuel [S], Grønbæk and Willis [GW], concerning the existence of bounded right approximate identities in the algebra of compact operators, enable us to involve a construction due to Willis [Wi] of Banach spaces that have the bounded compact approximation property, but fail to have the approximation property.

We also consider another quantitative problem. Recall that $R \in L(E,F)$ is strictly singular, denoted $R \in S(E,F)$, if the subspace $M \subset E$ is finite-dimensional whenever the restriction $R_{|M}$ defines an embedding into. The operator $R\colon E \to F$ is strictly cosingular, denoted $R \in P(E,F)$, if the subspace $N \subset F$ has finite codimension whenever $Q_N R$ is a surjection. Here $Q_N\colon F \to F/N$ is the quotient map. Schechter and Weis introduced

$$\Delta(T) = \sup_{M} \inf_{N \subset M} \|T_{|N}\|, \qquad \nabla(T) = \sup_{W} \inf_{V \supset W} \|Q_V T\|$$

for $T \in L(E,F)$. In the definitions M and N stand for closed infinite-dimensional subspaces of E, while W and V denote closed infinite-codimensional subspaces of F. These quantities occur for instance in perturbation theory, see [Sch], [SW] and [W1]. The seminorms Δ and ∇ measure the deviation of an operator from strict singularity or cosingularity in the sense that $\Delta(T) = 0$ if and only if $T \in S(E,F)$, $\nabla(T) = 0$ if and only if $T \in P(E,F)$. In addition,

$$\Delta(T) \le \operatorname{dist}(T, S(E,F)), \qquad \nabla(T) \le \operatorname{dist}(T, P(E,F))$$

for any $T \in L(E,F)$. It is a natural question (stated in [R] for a quantity uniformly comparable to Δ) to investigate whether Δ and ∇ are equivalent to the corresponding distance functions. We establish that this does not hold in general.

A common ingredient of these examples appears in the guise of a general connection between certain seminorms corresponding to injective or surjective operator ideals and associated approximation properties. Section 1 studies the general setting. Sections 2 and 3 contain the applications to the self-duality problem for the essential norm and to the comparability problem for measures of non-strict singularity and cosingularity.

1. Approximation properties and ideal variations

This section explains the general observation that provides a common starting point for our examples. We will use standard notation from Banach space theory in accordance with [LT2]. We refer [Pi] for the definition and for examples of (normed) operator ideals in the sense of Pietsch.

Let E be a Banach space, Q_E: $\ell^1(B_E) \to E$ the surjection defined by $(a_x) \to \sum_{x \in B_E} a_x x$ for $(a_x) \in \ell^1(B_E)$ and J_E: $E \to \ell^\infty(B_{E'})$ the isometric embedding $x \to (\langle x', x\rangle)_{x' \in B_{E'}}$. An operator ideal I is said to be injective if $I(E,F) = \{S \in L(E,F)\colon J_F S \in I(E, \ell^\infty(B_{F'}))\}$, and surjective if $I(E,F) = \{S \in L(E,F)\colon SQ_E \in I(\ell^1(B_E), F)\}$ for all Banach spaces E and F. The ideal K consisting of the compact operators is injective and surjective. A normed operator ideal I is closed if its components $I(E,F)$ are closed in the operator norm.

Let I be a normed operator ideal. The **outer I-variation** of $S \in L(E,F)$,

$$\gamma_I(S) = \inf\{\epsilon > 0\colon SB_E \subset RB_Z + \epsilon B_F, R \in I(Z,F),\ Z \text{ any Banach space}\},$$

was introduced by Astala [A]. The (Hausdorff) measure of non-compactness γ_K is well known. We define the **inner I-variation** of $S \in L(E,F)$ by

$$\begin{aligned}\beta_I(S) = \inf\{\epsilon > 0\colon &\text{ there is a Banach space } Z \text{ and } R \in I(E,Z)\\ &\text{such that } \|Sx\| \le \|Rx\| + \epsilon\|x\|, x \in E\}.\end{aligned}$$

For instance, $\beta_K(S) = \gamma_K(S')$ for $S \in L(E,F)$, [GMa, Thm. 2] (see also Theorem 2.3).

The quotient norm modulo the operator norm closure of the normed ideal I is $\|S\|_I = \mathrm{dist}(S, I(E,F))$ for $S \in L(E,F)$. The preceding quantities measure the deviation of an operator from the ideal I.

PROPOSITION 1.1: *Let I be a normed operator ideal and E, F Banach spaces.*

(i) *β_I and γ_I are submultiplicative seminorms on $L(E,F)$.*

(ii) *$\beta_I(S) = 0$ if and only if $J_F S$ is in the uniform closure of $I(E, l^\infty(B_{F'}))$.*

(iii) *$\gamma_I(S) = 0$ if and only if SQ_E is in the uniform closure of $I(l^1(B_E), F)$.*

(iv) *$\max\{\beta_I(S), \gamma_I(S)\} \leq \|S\|_I$ for any $S \in L(E,F)$.*

Proof: The properties of γ_I were verified in [A, 3.7, 3.11, 4.1]. The operators S with $\beta_I(S) = 0$ are identified by [J, 20.7.3]. It is a simple exercise to check the remaining facts concerning β_I. ∎

We determine under which conditions the ideal variations β_I or γ_I are comparable with the quotient norm $\|\cdot\|_I$ for closed injective or surjective operator ideals I. Special approximation conditions are required for this purpose. Let I be a closed injective operator ideal. The Banach space E is said to have the **injective I-approximation** property (abbreviated I-AP) provided there is a constant $c \geq 1$ such that

$$\inf\{\|R - RV\| : V \in I(E),\ \|\mathrm{Id} - V\| \leq c\} = 0 \tag{1.1}$$

for any Banach space Z and any $R \in I(E,Z)$. If I is a closed surjective operator ideal, then E has the **surjective I-approximation** property if there is a constant $c \geq 1$ so that

$$\inf\{\|R - VR\| : V \in I(E),\ \|\mathrm{Id} - V\| \leq c\} = 0 \tag{1.2}$$

for any Banach space Z and any $R \in I(Z,E)$. Other approximation properties associated with operator ideals have appeared in [Re] and [GW]. The injective I-AP is also motivated by Banach algebra concepts. We refer to sections 2 and 3 for further discussions and examples.

THEOREM 1.2: *Suppose that I is a closed injective operator ideal. Then the Banach space E has the injective I-approximation property if and only if the seminorms β_I and $\|\cdot\|_I$ are equivalent on $L(E,F)$ for all Banach spaces F.*

Proof: Assume that E has the injective I-AP with constant c and let $S \in L(E,F)$. If $\mu > \beta_I(S)$, then there is a Banach space Z and $R \in I(E,Z)$ so that

$\|Sx\| \leq \|Rx\| + \mu\|x\|$ whenever $x \in E$. By assumption there is for any $\epsilon > 0$ an operator $V \in I(E)$ satisfying $\|R - RV\| < \epsilon$ and $\|\operatorname{Id} - V\| \leq c$. Hence

$$\|Sx - SVx\| \leq \|R(x - Vx)\| + \mu\|x - Vx\| \leq \epsilon\|x\| + \mu\|\operatorname{Id} - V\|\|x\| \leq (c\mu + \epsilon)\|x\|$$

for $x \in E$. This yields $\|S\|_I \leq c\mu + \epsilon$ since $SV \in I$, and altogether

$$\beta_I(S) \leq \|S\|_I \leq c\beta_I(S).$$

The reverse implication is based on a renorming argument. Suppose that E fails to have the injective I-approximation condition. We construct a Banach space F so that β_I and $\|\cdot\|_I$ are not equivalent on $L(E, F)$. The assumption implies according to (1.1) that there are (for any $n \in \mathbb{N}$) Banach spaces Z_n, operators $R_n \in I(E, Z_n)$ and $\epsilon_n > 0$ with the following property:

$$\text{if } V \in L(E) \text{ satisfies } \|R_n - R_nV\| \leq \epsilon_n \text{ and } \|\operatorname{Id} - V\| \leq n, \quad \text{then } V \notin I(E). \tag{1.3}$$

We may suppose that $\epsilon_n = 1$ by passing to $(1/\epsilon_n)R_n$. Equip E with the equivalent norm

$$|x|_n = \frac{1}{n}\|x\| + \|R_nx\|, \quad x \in E,$$

and consider the identity operator T_n: $E \to F_n = (E, |\cdot|_n)$, $T_nx = x$. Clearly $\beta_I(T_n) \leq 1/n$ for $n \in \mathbb{N}$.

Assume next that $V \in L(E, F_n)$ satisfies $\|T_n - V\| \leq 1$. Hence, by viewing V also as an operator $E \to E$ in the obvious fashion, one obtains

$$1 \geq \|T_n - V\| = \sup_{x \in B_E} \Big(\frac{1}{n}\|x - Vx\| + \|R_n(x - Vx)\|\Big) \geq \|R_n - R_nV\|$$

and

$$\|\operatorname{Id} - V\| \leq n \sup_{x \in B_E} |T_nx - Vx|_n \leq n.$$

Condition (1.3) implies that $V \notin I(E, F_n)$ and it follows that $\|T_n\|_I \geq 1$. Finally, let F denote the ℓ^2 – sum $(\bigoplus_{n \in \mathbb{N}} F_n)_{\ell^2}$ and set $S_n = J_nT_n \in L(E, F)$, where J_n: $F_n \to F$ is the inclusion map. It is easily verified that

$$\|S_n\|_I = \|T_n\|_I \geq 1, \quad \beta_I(S_n) = \beta_I(T_n) \leq \frac{1}{n}$$

for $n \in \mathbb{N}$. This completes the proof. ∎

The corresponding result for a surjective ideal extends the previously known compact and weakly compact cases [AT1, 2.3, 2.5], [AT2, Thm. 1]. The argument is quite similar, but details are included for completeness.

THEOREM 1.3: *Suppose that I is a closed surjective operator ideal. Then the Banach space F has the surjective I-approximation property if and only if the seminorms γ_I and $\|\cdot\|_I$ are equivalent on $L(E,F)$ for all Banach spaces E.*

Proof: Suppose that F has the surjective I-AP with constant c and let $S \in L(E,F)$. If $\gamma_I(S) < \mu$, then there is a Banach space Z and $R \in I(Z,F)$ with $\mathrm{S}B_E \subset \mathrm{R}B_Z + \mu B_F$. Let $\epsilon > 0$. By assumption there is $U \in I(F)$ such that $\|R - UR\| < \epsilon$ and $\|\operatorname{Id} - U\| \leq c$. One has $\|S\|_I \leq \|S - US\|$. Moreover, for any $x \in B_E$ there is $z \in B_Z$ so that $\|Sx - Rz\| \leq \mu$ and we obtain

$$\|Sx - USx\| \leq \|\operatorname{Id} - U\|\|Sx - Rz\| + \|Rz - URz\| < c\mu + \epsilon.$$

Hence $\gamma_I(S) \leq \|S\|_I \leq c\gamma_I(S)$ for all $S \in L(E,F)$.

Assume towards the converse implication that F does not have the surjective I-AP. It suffices to construct a Banach space E such that γ_I and $\|\cdot\|_I$ fail to be comparable on $L(E,F)$. Fix $n \in \mathbb{N}$. Since F does not have the surjective I-AP there exist by (1.2) Banach spaces Z_n, operators $R_n \in I(Z_n, F)$ as well as $\epsilon_n > 0$ so that

$$\text{if } V \in L(F) \text{ satisfies } \|R_n - VR_n\| \leq \epsilon_n \text{ and } \|\operatorname{Id} - V\| \leq n, \quad \text{then } V \notin I(F). \tag{1.4}$$

We may normalize to $\epsilon_n = 1$ by considering $(1/\epsilon_n)R_n$. Set

$$B_n = \frac{1}{n}B_F + \overline{R_n B_{Z_n}}$$

and consider the equivalent norm $|x|_n = \inf\{t \geq 0\colon x \in tB_n\}$ on F. Let T_n be the identity mapping from $E_n = (F, |\cdot|_n)$ to F. By definition $\gamma_I(T_n) \leq 1/n$ for $n \in \mathbb{N}$. We claim that $\|T_n\|_I \geq 1$, $n \in \mathbb{N}$.

In fact, suppose that $V \in L(E_n, F)$ and $\|T_n - V\| \leq 1$. One obtains (by simultaneously viewing V as an operator on F) that

$$\|R_n - VR_n\| = \sup_{z \in B_{Z_n}} \|R_n z - VR_n z\| \leq \sup_{x \in B_n} \|x - Vx\| \leq 1$$

and

$$\|\operatorname{Id} - V\| = \sup_{x \in B_F} \|x - Vx\| \leq n\|T_n - V\| \leq n.$$

Conclude from (1.4) that $V \notin I(E_n, F)$ so that $\|T_n\|_I \geq 1$. Finally, let $E = (\bigoplus_{m \in \mathbb{N}} E_m)_{\ell^2}$ and define $S_n \in L(E,F)$ by $S_n(x_m) = T_n x_n$ for $(x_m) \in E$. One obtains

$$\|S_n\|_I \geq 1, \quad \gamma_I(S_n) \leq \frac{1}{n}, \quad n \in \mathbb{N}. \qquad \blacksquare$$

2. The essential norm is not self-dual

This section constructs Banach spaces E and F so that (0.1) does not hold in $L(E,F)$ for any $c > 0$. This solves negatively the self-duality problem stated in the introduction. We commence by recalling some standard approximation properties.

Let E be a Banach space. E has the approximation property (AP in short), if for any $\epsilon > 0$ and any compact subset $D \subset E$, there is a finite rank operator $R \in L(E)$ such that

$$\sup\{\|x - Rx\|: x \in D\} < \epsilon. \tag{2.1}$$

E has the bounded approximation property (BAP) if there is a constant $c \geq 1$ such that for any $\epsilon > 0$ and any compact subset $D \subset E$ we find a finite rank operator $R \in L(E)$ so that $\|R\| \leq c$ and (2.1) holds. Finally, E has the bounded compact approximation property (BCAP) if one allows compact operators $R \in L(E)$ with $\|R\| \leq c$ in the preceding condition. We refer to [LT2, 1.e,2.d], [LT3, 1.g] for a comprehensive study of these properties.

A bounded net (x_α) of a Banach algebra A is a bounded left (respectively right) approximate identity in A if $\lim_\alpha x_\alpha x = x$ (resp., $\lim_\alpha x x_\alpha = x$) for all $x \in A$. The existence of bounded left or right approximate identities in the algebra $K(E)$ was investigated in [Di], [GW] and [S]. It is easily checked that the surjective K-AP of section 1 equals the BCAP. Dixon [Di, 2.6] observed that $K(E)$ admits a bounded left approximate identity if and only if E has the BCAP. We need a similar characterization of the injective K-AP. It connects, through Theorem 1.2, bounded right approximate identities in $K(E)$ with the behaviour of the seminorm β_K on $L(E,F)$ for all Banach spaces F.

PROPOSITION 2.1: *A Banach space E has the injective K-AP if and only if $K(E)$ admits a bounded right approximate identity.*

Proof: Suppose that E has the injective K-AP. Thus we find a constant $c \geq 1$ such that there is $U \in K(E)$ satisfying $\|U\| \leq c$ and $\|R - RU\| < \epsilon$ for any given $R \in K(E)$ and $\epsilon > 0$. This implies that there is a bounded net (U_α) in $K(E)$ with $\lim_\alpha RU_\alpha = R$ for all $R \in K(E)$, see [BD, p. 58].

Assume that $K(E)$ admits a bounded right approximate identity (U_α) with bound c. Suppose that Z is a Banach space, $R \in K(E,Z)$ and $\epsilon > 0$. We claim that there is $U \in K(E)$ with uniformly bounded norm and $\|R - RU\| < \epsilon$.

There is no loss of generality in assuming $Z = l^\infty(I)$ for some index set I. Hence there is a finite rank operator $R_o\colon E \to l^\infty(I)$, $\|R - R_o\| < (3(c+1))^{-1}\epsilon$, since $l^\infty(I)$ has the approximation property. Consider the finite rank operator $S = \|R_o\|P \in L(E)$, where P is a projection onto a direct complement of $\mathrm{Ker}R_o$ (the kernel $\mathrm{Ker}R_o$ has finite codimension in E). Clearly $\|R_o x\| \le \|Sx\|$ for all $x \in E$. By assumption there is a compact operator $U \in K(E)$ with $\|U\| \le c$ and $\|S - SU\| < \epsilon/3$. Thus $\|R_o - R_o U\| < \epsilon/3$ from the choice of S and

$$\begin{aligned}\|R - RU\| &\le \|R - R_o\| + \|R_o - R_o U\| + \|R_o U - RU\| \\ &< (3(c+1))^{-1}\epsilon + \epsilon/3 + (3(c+1))^{-1}\epsilon\|U\| < \epsilon. \qquad \blacksquare\end{aligned}$$

We also require a recent characterization due to Samuel [S, Thm. 1], Grønbæk and Willis [GW, 2.6].

THEOREM 2.2: *Let E be a Banach space. Then $K(E)$ admits a bounded right approximate identity if and only if E' has the following approximation property:*

(2.2) *there is a constant $c \ge 1$ so that for any finite set $D \subset E'$ and $\epsilon > 0$ there is $R \in K(E)$ satisfying $\|R\| \le c$ and $\|R'x' - x'\| < \epsilon$ for all $x' \in D$.*

Proof: We recall for completeness the argument from [S] for the implication from left to right (the other implication is not used here). Suppose that $K(E)$ has a bounded right approximate identity, let $D \subset E'$ be a finite set and $\epsilon > 0$. Set $C = \max\{\|x'\|\colon x' \in D\}$. The principle of local reflexivity implies that there is a finite rank projection P on E so that $D \subset \mathrm{Im}(P')$, see [JRZ, Cor. 3.2]. The assumption gives a uniform constant c and $R \in K(E)$ satisfying $\|R\| \le c$ and $\|P - PR\| < \epsilon/C$. Hence

$$\|x' - R'x'\| = \|P'x' - R'P'x'\| \le C\|P - PR\| < \epsilon$$

for all $x' \in D$. $\blacksquare$

The quantities γ_K and β_K are known to be uniformly self-dual. (2.3) below is due to Goldenstein and Markus [GM, Thm. 3] (see also [A, 5.9]) and (2.4) is [GMa, Thm. 2]. Proofs are supplied for the sake of completeness at the suggestion of the referee, since [GM] is not easily available.

THEOREM 2.3: *If E and F are Banach spaces and $S \in L(E,F)$, then*

(2.3) $\frac{1}{2}\gamma_K(S) \le \gamma_K(S') \le 2\gamma_K(S)$,

(2.4) $\beta_K(S) = \gamma_K(S')$

and $\frac{1}{2}\beta_K(S) \leq \beta_K(S') \leq 2\beta_K(S)$.

Proof: Suppose that $\alpha > \gamma_K(S)$. There are elements $z_1, ..., z_n$ of F so that $SB_E \subset \{z_1, ..., z_n\} + \alpha B_F$. Fix $\epsilon > 0$, set $R = \max\{\|z_i\|: i = 1, \ldots, n\}$ and partition $\{\lambda \in \mathbb{K}: |\lambda| \leq R\}$ into a finite union of sets B_j, $j = 1, \ldots, m$, where the diameter $\delta(B_j) < \epsilon$ for all j. Put

$$D(r_1, \ldots, r_n) = \{x' \in B_{F'}: \langle x', z_i \rangle \in B_{r_i} \text{ for } i = 1, \ldots, n\}$$

for each $(r_1, \ldots, r_n) \in \{1, \ldots, m\}^n$. Clearly the family

$$\{D(r_1, \ldots, r_n): (r_1, \ldots, r_n) \in \{1, \ldots, m\}^n\}$$

is a finite partition of $B_{F'}$. We estimate the diameter $\delta(S'D(r_1, \ldots, r_n))$ for non-empty $D(r_1, \ldots, r_n)$. Let $x', y' \in D(r_1, \ldots, r_n)$, $x \in B_E$ and pick $i \in \{1, \ldots, n\}$ so that $\|Sx - z_i\| \leq \alpha$. Hence

$$\begin{aligned} |\langle S'x' - S'y', x \rangle| &= |\langle x', Sx \rangle - \langle y', Sx \rangle| \\ &\leq |\langle x', Sx - z_i \rangle| + |\langle x', z_i \rangle - \langle y', z_i \rangle| + |\langle y', z_i - Sx \rangle| \\ &\leq 2\alpha + \epsilon \end{aligned}$$

since $\langle x', z_i \rangle, \langle y', z_i \rangle \in B_{r_i}$. Thus $\|S'x' - S'y'\| \leq 2\alpha + \epsilon$ and $\delta(S'D(r_1, \ldots, r_n)) \leq 2\alpha + \epsilon$. Deduce that $\gamma_K(S') \leq 2\gamma_K(S)$. The other inequality $\gamma_K(S) \leq 2\gamma_K(S')$ of (2.3) is established in a completely symmetric manner.

Suppose next that $\alpha > \beta_K(S)$ and take a space Z and $R \in K(E, Z)$ so that $\|Sx\| \leq \|Rx\| + \alpha\|x\|$ for all $x \in E$. The geometric Hahn–Banach separation theorem yields $S'B_{F'} \subset R'B_{Z'} + \alpha B_{E'}$ and thus $\gamma_K(S') \leq \alpha$. Deduce that $\gamma_K(S') \leq \beta_K(S)$.

Assume that $\alpha > \gamma_K(S')$. There is a Banach space Z and $R \in K(Z, E')$ so that $S'B_{F'} \subset RB_Z + \alpha B_{E'}$. The Hahn–Banach theorem implies again that $\|S''z\| \leq \|R'z\| + \alpha\|z\|$ for all $z \in E''$ and hence $\beta_K(S'') \leq \alpha$. One obtains

$$\beta_K(S) = \beta_K(K_F S) \leq \beta_K(S'') \leq \gamma_K(S'),$$

where K_F stands for the canonical inclusion $F \to F''$. This establishes (2.4) and the estimates of $\beta_K(S')$ are obvious from (2.3) and (2.4). ∎

The theorem below contains some of the known positive results concerning the self-duality of the essential norm [AJS, Thm. 3], [D, Prop. 3].

THEOREM 2.4: *Let E and F be Banach spaces. There is a constant $c > 0$ such that*

$$c\|S\|_K \leq \|S'\|_K \leq \|S\|_K, \quad S \in L(E,F),$$

provided one of the following conditions is satisfied:

(i) *E has the injective K-AP,*

(ii) *F has the BCAP,*

(iii) *there is a projection $P\colon F'' \to F$.*

Proof: Suppose that E has the injective K-AP with constant c. The estimates

$$\|S\|_K \leq c\beta_K(S) = c\gamma_K(S') \leq c\|S'\|_K$$

are seen from Theorem 1.2 and (2.4). Next, let F have the BCAP with constant c. Thus

$$\|S\|_K \leq (c+1)\gamma_K(S) \leq 2(c+1)\gamma_K(S') \leq 2(c+1)\|S'\|_K,$$

$S \in L(E,F)$, in view of [LS, 3.6] and (2.3). Part (iii) is well known, see [AJS, p. 160]. ∎

We are ready for the promised example, where Theorem 1.2 plays a crucial role. The existence of Banach spaces that have the BCAP, but fail to have the AP, is also essential. Such spaces were recently obtained by Willis [Wi].

Example 2.5: There are Banach spaces E and F as well as a sequence (S_n) of $L(E,F)$ so that $\|S_n\|_K = 1$ for all $n \in \mathbb{N}$, but $\|S_n'\|_K \to 0$ as $n \to \infty$.

Proof: Let X be the separable reflexive Banach space constructed by Willis [Wi, Prop. 3 and 4], such that its dual X' has the BCAP but fails to have the AP. A construction due to James and Lindenstrauss (see [LT2, 1.d.3] and [L]) yields a Banach space Z, so that Z'' has a Schauder basis and Z''/Z is isomorphic to X. We require the following properties of Z.

CLAIM: Z''' has the surjective K-AP, but Z'' does not have the injective K-AP.

This fact is essentially proved in [GW, 4.3], but we reproduce the argument for convenience. First of all, Z''' has the surjective K-AP (that is, the BCAP), since Z''' is isomorphic to $Z' \oplus X'$ and Z' has the BAP (because Z'' has a basis). On the other hand, Z''' fails to have property (2.2). Indeed, $K(Z'') = A(Z'')$ and thus otherwise (2.2) would imply that Z''' has the AP. Here $A(Z'')$ denotes the uniform closure in $L(Z'')$ of the class of finite rank operators on Z''. This is not possible in view of the choice of X', since there is a complemented copy of X' in Z'''. Consequently Proposition 2.1 and Theorem 2.2 imply that Z'' does not have the injective K-AP.

Set $E = Z''$. The proof of Theorem 1.2 provides equivalent renormings $F_n = (E, |\cdot|_n)$ of E as well as a sequence (S_n) of $L(E,F)$, where $F = (\bigoplus_{n\in\mathbb{N}} F_n)_{\ell^2}$, such that

$$\beta_K(S_n) \leq 1/n, \quad \|S_n\|_K = 1$$

for all $n \in \mathbb{N}$. The dual E' has the surjective K-AP according to the Claim and Theorem 1.3 gives a uniform constant $c > 0$ so that by (2.4)

$$\|S_n'\|_K \leq c\gamma_K(S_n') = c\beta_K(S_n) \leq c/n$$

for $n \in \mathbb{N}$. ∎

Remarks: (i) Let E, F and S_n be as in the preceding example and equip $E \oplus F$ with the norm $\|(x,y)\| = \|x\| + \|y\|$. Then we get for $U_n \in L(E\oplus F)$, $U_n(x,y) = (0, S_n x)$, that $\|U_n\|_K = \|S_n\|_K = 1$ for $n \in \mathbb{N}$ and $\|U_n'\|_K = \|S_n'\|_K \to 0$ as $n \to \infty$.

(ii) The injective K-approximation property of a Banach space E (equivalently, the existence of a bounded right approximate identity in $K(E)$) admits no obvious general characterization in terms of standard approximation properties. For instance, E' has the BCAP by (2.2) whenever E has the injective K-AP, but the fact that the converse does not hold is crucial for Example 2.5 (see also [GW, 4.3]). This example also answers Problem 2.9 of [AT1] negatively. If E' has the BAP (for instance, E has a shrinking basis), then E has the injective K-AP (cf. [GW, 3.3]). The reflexive space X of [Wi, Prop. 4] has the injective K-AP, but X' does not have the BAP. Moreover, above Z'' has a Schauder basis but Z'' fails to have the injective K-AP.

3. Measures of non-strict singularity and cosingularity

This section finds Banach spaces E and F such that the quantities Δ and ∇ (defined in the introduction) fail to be comparable with the corresponding quotient norms $\|\cdot\|_S$ and $\|\cdot\|_P$ on $L(E,F)$. The strictly singular operators S form an injective ideal and the strictly cosingular operators P a surjective ideal. We compare Δ and ∇ to the ideal variations β_S and γ_P in order to apply Theorems 1.2 and 1.3.

LEMMA 3.1: *Let E and F be Banach spaces. Then*

$$\Delta(T) \leq \beta_S(T), \ \nabla(T) \leq \gamma_P(T),$$

for $T \in L(E,F)$.

Proof: Suppose that $\lambda > \beta_S(T)$ and $\epsilon > 0$. There is a Banach space Z and a strictly singular operator $R: E \to Z$ such that $\|Tx\| \leq \|Rx\| + \lambda\|x\|$, $x \in E$. Let M be an arbitrary closed infinite-dimensional subspace of E. Hence there is an infinite-dimensional subspace $N \subset M$ with $\|R_{|N}\| < \epsilon$, see [Pi, 1.9.1]. Thus $\inf_{L \subset M} \|T_{|L}\| \leq \|T_{|N}\| \leq \lambda + \epsilon$. This yields the first inequality.

Suppose that $\lambda > \gamma_P(T)$ and assume that $R \in P(Z,E)$ satisfies $TB_E \subset RB_Z + \lambda B_F$. If $W \subset F$ is a closed infinite-codimensional subspace and $\epsilon > 0$, then there is a closed infinite-codimensional subspace V of F, $V \supset W$, so that $\|Q_V R\| < \epsilon$ [Pi, 1.10.1]. Let $x \in B_E$. There are $z \in B_Z$ and $v \in B_F$ satisfying $Tx = Rz + \lambda v$. Consequently $\inf_{L \supset W} \|Q_L T\| \leq \|Q_V T\| \leq \lambda + \epsilon$. This implies the second inequality. ∎

Examples of Banach spaces to which the following corollary applies are given in Theorem 3.6.

COROLLARY 3.2: *Let E be a Banach space.*

(i) *If E does not have the injective S-AP, then there is a Banach space F such that Δ and $\|\cdot\|_S$ fails to be comparable on $L(E,F)$.*
(ii) *If E does not have the surjective P-AP, then there is a Banach space F such that ∇ and $\|\cdot\|_P$ fails to be comparable on $L(F,E)$.*

Proof: Combine the preceding lemma with Theorems 1.2 and 1.3. ∎

Theorems 1.2 and 1.3 have similar consequences for some other measures of non-strict-singularity and cosingularity used in the literature [Sch], [Z]. Let $j(S) = \inf\{\|Sx\|: x \in E, \|x\| = 1\}$ and $q(S) = \sup\{\delta > 0: \delta B_F \subset SB_E\}$ denote the injection, respectively the surjection modulus of $S \in L(E,F)$. Set

$$u(R) = \sup\{j(R_{|M}): M \subset E \text{ is infinite-dimensional}\},$$
$$v(R) = \sup\{q(Q_V R): V \text{ has infinite codimension in } F\}$$

for $R \in L(E,F)$. It is evident that $u(R) \leq \Delta(R)$ and $v(R) \leq \nabla(R)$ for any $R \in L(E,F)$, so that Corollary 3.2 remains valid for u and v.

These quantities have been computed for operators on some concrete Banach spaces:

$$\begin{aligned} u(R) &= \Delta(R) = \beta_S(R) = v(R) \\ &= \nabla(R) = \gamma_P(R) = \|R\|_K = \lim_{n\to\infty} \|Q_n R Q_n\| \end{aligned} \tag{3.1}$$

for $R \in L(l^p)(1 \leq p < \infty)$ or $R \in L(c_o)$. These equalities follow from [W2, 2.2] and [B, Chapter 2.2] combined with 3.1 for β_S and γ_P. Above Q_n are the natural projections $Q_n(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=n}^{\infty} a_i e_i$ on l^p or c_o, and (e_i) is the standard unit vector basis. In addition,

$$\Delta(R) = \beta_S(R) = \nabla(R) = \gamma_P(R) = \|R\|_S = \|R\|_P = \lim \sup_{|A|\to 0} \|\chi_A R\| \tag{3.2}$$

when $R \in L(L^1(0,1))$, [W2, 6.1], [W3]. Here $|A|$ denotes the (Lebesgue) measure of a measurable set $A \subset [0,1]$ and χ_A is the characteristic function.

We conclude by verifying that many classical Banach spaces fail to have the injective S-approximation and the surjective P-approximation properties. These examples illustrate some differences between the general I-AP and standard approximation properties, for instance the class of Banach spaces that does not satisfy such an I-approximation property is quite extensive for many ideals I. We require several preliminary observations. Let E and F be Banach spaces. V denotes the closed operator ideal consisting of the completely continuous operators (thus $S \in V(E,F)$ if (Sx_n) converges in norm for all weakly convergent sequences (x_n) of E). Suppose that A and B are operator ideals. $S \in L(E,F)$ belongs to the quotient $A \circ B^{-1}(E,F)$ if $SR \in A(Z,F)$ whenever $R \in B(Z,E)$, Z an arbitrary Banach space. Similarly, $S \in B^{-1} \circ A(E,F)$ if $RS \in A(E,Z)$ for all $R \in B(F,Z)$.

Proposition 3.3:

(i) *Let I be a closed surjective operator ideal and E a Banach space. If $Id_E \notin V \circ I^{-1}(E)$ and $I(E) \subset V \circ I^{-1}(E)$, then E fails to have the surjective I-AP.*

(ii) *Let I be a closed injective operator ideal. If $\mathrm{Id}_E \notin I^{-1} \circ V(E)$ and $I(E) \subset I^{-1} \circ V(E)$, then E fails to have the injective I-AP.*

Proof: (i) $\mathrm{Id}_E \notin V \circ I^{-1}(E)$ implies that there exists a Banach space Z and a non-completely continuous operator $R \in I(Z,E)$. There is a weakly convergent sequence (x_n) in Z as well as $c > 0$ so that $\|Rx_n - Rx_m\| \geq c$ whenever $n \neq m$. Set $z_j = x_{2j+1} - x_{2j}$ for $j \in \mathbb{N}$. Then (z_j) is a weak null-sequence of Z and $\|Rz_j\| \geq c$ for all $j \in N$. Suppose without loss of generality (pass to λR for some $\lambda > 0$) that $\|z_j\| \leq 1$ and $\|Rz_j\| \geq 1$ for $j \in \mathbb{N}$. Let $V \in I(E)$ be arbitrary. Hence $VR \in V(Z, E)$ by the assumption on I and

$$\|R - VR\| \geq \|Rz_j - VRz_j\| \geq 1 - \|VRz_j\| > 1/2$$

for all sufficiently large j, since $\|VRz_j\| \to 0$ as $j \to \infty$. Thus E fails to have the surjective I-AP. The proof of part (ii) is similar. ∎

Suppose that I is a closed injective operator ideal. Then the components

$$I^*(E, F) = \{S \in L(E, F) \colon S' \in I(F', E')\},$$

E and F Banach spaces, define a closed surjective operator ideal I^*.

Proposition 3.4: *Assume that I is a closed injective operator ideal so that $R \in I(E, F)$ implies $R'' \in I(E'', F'')$ for all Banach spaces E and F. If the Banach space E has the injective I-AP, then E' has the surjective I^*-AP.*

Proof: It is enough to consider $R \in I^*(l^1(K), E')$ in condition (1.2), for arbitrary index sets K, in view of the surjectivity of I^*. Let $K_E \colon E \to E''$ be the canonical embedding. Thus $R'K_E \in I(E, l^\infty(K))$. Let $\epsilon > 0$. The injective I-AP of E provides a uniform constant c as well as $V \in I(E)$ so that $\|R'K_E - R'K_E V\| < \epsilon$ and $\|\mathrm{Id} - V\| \leq c$. One obtains that $V' \in I^*(E')$ by the assumption on I and

$$\|R - V'R\| \leq \|(K_E)'R''_{|l^1(K)} - V'(K_E)'R''_{|l^1(K)}\| \leq \|R'K_E - R'K_E V\| < \epsilon,$$

since $(K_E)'$ defines a norm-1 projection of E''' onto E'. ∎

Recall that $R \in L(E, F)$ is a weakly compact operator, denoted $R \in W(E, F)$, if RB_E is relatively weakly compact in F. The surjective W-AP was applied in [AT2]. A Banach space E has the Schur property, if the weakly convergent sequences of E are norm-convergent. $l^1(K)$ are among the spaces with this property for all index sets K. We refer to [LT1, II.5.b] for the definitions of $\mathcal{L}^1$- and $\mathcal{L}^\infty$-spaces. Standard examples of $\mathcal{L}^1$-spaces are $L^1(0,1)$, l^1 and $C(0,1)'$, while $C(K)$-spaces and $L^\infty(0,1)$ are $\mathcal{L}^\infty$-spaces.

PROPOSITION 3.5: *Let E be a $\mathcal{L}^1$- or a $\mathcal{L}^\infty$-space.*

(i) *E has the surjective W-AP if and only if E has the Schur property.*
(ii) *E has the injective W-AP if and only if E' has the Schur property.*

Proof: Part (i) is [AT2, Cor. 3]. Suppose towards (ii) that E has the injective W-AP. Thus E' has the surjective W-AP by Proposition 3.4 and Gantmacher's theorem. Hence E' has the Schur property in view of (i), since E' is also a $\mathcal{L}^1$- or a $\mathcal{L}^\infty$-space [LT1, II.5.7]. Conversely, assume that E' has the Schur property. Hence $W(E,Z) = K(E,Z)$ for all Banach spaces Z according to the duality properties of (weakly) compact operators and the injective W-AP coincides with the injective K-AP for such spaces E. Observe that E has here the injective K-AP by Proposition 2.1 and [GW, 3.3], since E' has the BAP [LT1, II.5.9]. ∎

Remark: [BD, p. 58] implies that the algebra $I(E)$ admits a bounded left (resp. right) approximate identity whenever E has the surjective (resp. the injective) I-AP. Simple examples demonstrate that the converse implication does not hold for arbitrary ideals I. For instance, c_0 fails to have the surjective W-AP in view of 3.5.i, but $W(c_o) = K(c_o)$ has a bounded left approximate identity. ℓ^1 does not have the injective W-AP by 3.5.ii, but $W(\ell^1) = K(\ell^1)$ admits a bounded right approximate identity in view of [GW, 3.3].

The following list concerning the injective S-AP and the surjective P-AP of some classical Banach spaces should be compared with Corollary 3.2 as well as (3.1), (3.2).

THEOREM 3.6:

(i) *Assume that the Banach space E contains a complemented copy of ℓ^p (some p, $1 \le p \le \infty$). Then E fails to have the injective S-AP. $C(X)$ has the injective S-AP whenever X is a countable compact metric space, but $C(0,1)$ does not have it.*
(ii) *Assume that E contains a complemented copy of ℓ^p (some p, $1 < p \le \infty$) or of c_o. Then E fails to have the surjective P-AP. ℓ^1 has the surjective P-AP, but $L^1(0,1)$ does not have it.*

Proof: (i) ℓ^p fails to have the injective S-AP for $1 \le p < \infty$. This is a consequence of Proposition 3.3(ii) if $1 < p < \infty$. In fact, $\mathrm{Id}_{\ell^p} \notin (S^{-1} \circ V)(\ell^p)$, since the inclusion map $J\colon \ell^p \to \ell^q$ is strictly singular, but not completely continuous, for $1 < p < q < \infty$ (ℓ^p and ℓ^q are totally incomparable). Also,

$S(l^p) = K(l^p) \subset S^{-1} \circ V(l^p)$, as $K(l^p)$ is the unique proper closed ideal of $L(l^p)$ [Pi, 5.1–5.2].

The claims for $E = C(X)$, l^∞, $C(0,1)$ and l^1 are seen from Proposition 3.5(ii). We simply list below the facts that reduce our claim to the case of the injective W-AP.

(a) $S(l^1) = K(l^1)$ and $K(l^1, Z) \subsetneqq W(l^1, Z) \subset S(l^1, Z)$ for any Z without the Schur property,

(b) $S(l^\infty, Z) = W$ for any Z, see [LT2, 2.f.4],

(c) $S(C(0,1), Z) = W$ for any Z according to a result of Pelczynski [P, p. 35].

(d) $S(C(X), Z) = W = K$ for any Z by [P, p. 35] and duality, since $C(X)' = l^1$ if X is a countable compact metric space.

Finally, if E is a Banach space containing a complemented copy of l^p for some p, $1 \leq p \leq \infty$, then E also fails to have the injective S-AP, since this property is inherited by complemented subspaces.

(ii) It suffices to verify towards the first claim that l^p $(1 < p < \infty)$ and c_o does not have the surjective P-AP. Total incomparability implies that the natural inclusion map $J \in P(l^r, l^p) \sim V(l^r, l^p)$ for $1 < r < p < \infty$, and thus $\mathrm{Id}_{l^p} \notin (V \circ P^{-1})(l^p)$. Similarly, $\mathrm{Id}_{c_o} \notin (V \circ P^{-1})(c_o)$. Moreover, $P(E) = K(E) \subset V \circ P^{-1}(E)$ for $E = l^p$ or c_o, [Pi, 5.1–5.2]. Thus Proposition 3.3(i) implies the claim in these cases.

In addition, $P(Z, L^1(0,1)) = W$ and $P(Z, l^1) = K = W$ for any Banach space Z, [P, p.39]. The claim follows from Proposition 3.5(i) in these cases. Finally, any embedding of l^2 into l^∞ is strictly cosingular and $P(l^\infty) = W(l^\infty)$ according to the Dunford–Pettis property of l^∞ and [LT2, 2.f.4]. The failure of the surjective P-AP for l^∞ is thus verified by imitating the proof of Proposition 3.3(i). ∎

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